ON A THREE-SERVER QUEUE WITH CONSULTATION BY MAIN SERVER CONTROLLED BY AN UPPER BOUND ON NUMBER OF INTERRUPTIONS

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Abstract--*This paper analyses a queueing model with three servers, namely a main server and two identical regular servers. The main server not only serves customers but also provides consultation to the regular servers with a preemptive priority over customers. The customersat the main server undergo interruptions during their service. The consultations are limited by an upper bound on the number of interruptions to a customer at the main server. The arrivals to the system and requirement of consultation follow independent Poisson processes. The service times at the main server and the regular servers are assumed to follow mutually independent phase type distributions. The stability of the system is established. Some performance measures are studied numerically.*

Keywords: *main server, regular server, consultation, interruption*

Introduction[1]

In a multi-server queueing system providing same type of services, some of the servers (trainees or less experienced ones) need clarifications or help frequently. So an experienced server provides timely clearances together with serving customers. These are seen to occur in banks (with the manager in addition to providing service to customers, helping other bank staff in their work also), hospitals (where the chief physician treats patients and clarifies the doubts of the fellow doctors), super markets, etc.

Chakravarthy [4] introduced a multi-server queueing system with consultations. There are 'c' servers. One of these 'c' servers are referred to as the main server and the others as the regular servers. The main server provides preemptive priority over customers to the regular servers on FIFO basis for consultation. Thus the service of the customer at the main server will be interrupted when a consultation occurs. The service of the interrupted customer at the main server will be resumed after all consultations are completed. The regular servers receive any number of consultations during the service of a customer. The service times are exponentially distributed with mean μ_1 at the main server and μ_2 at the identical regular servers. Queueing system with consultation has many applications in daily life. One such example is given in the above mentioned work.

Krishnamoorthy et.al [10] discussed a single server queueing model with interruptions to the server controlled by a finite number of interruptions and a super clock. When the number of interruptions already befell to the server reaches the upper bound, no further interruptions are allowed to the customer being served. A super clock is also there to limit the number of interruptions.

Queues with service interruptions was first studied by White and Christie [16] with exponentially distributed interruption duration. At the end of an interruption the service will be resumed. Some of the earlier papers which analyse queueing models with service interruptions, assuming general distributions for the service and interruption durations, are by Gaver [7], Keilson [9], Avi-Izhak and Naor [1] and Fiems et. al [6].

In the present paper we consider a multi-server queueing system equipped with three servers. Here customers arrive according to a Poisson process. An arriving customer enters into service immediately if at least one server is free, else joins a queue. The customer will be served by the main server whenever the main server and at least one of the regular servers is free. If both the regular servers are idle and the main server is busy, then the customer will approach any one of the regular server with probability $\frac{1}{2}$.

The main server offers consultation to the regular servers whenever it is necessary. Requirement of consultation arises according to a Poisson process with rate θ_i, where i is the number of busy regular servers, i = 1,2. When both the regular severs need consultation, a queue is formed for consultation and it is provided in FIFO basis. In order to distinguish the regular servers, we denote them $\Re_1$ and $\Re_2$. Duration of consultation follows exponential distribution. After getting consultation, the regular servers resume the services at the phases where they were suspended.

The service of the customer at the main server is said to be interrupted when the regular server requires a consultation. The main server immediately attends the request for consultation by the regular server. The customer at the main server has to wait until the completion of the consultation to get his service completed. After the interruption (consultation) completion, the main server resume the service at the phase where it was suspended. It is not fair to interrupt a customer at the main server infinitely many times. So we impose an upper bound M to the number of interruptions to the customer at the main server. If the number of interruptions already befell on the customer at the main server has reached M, then the main server will complete the present service before attending any more consultation. At this time the regular server has to wait to get the consultation.

Queueing Model[2]

The arrivals to the system follows Poisson process with rate λ. The service times at the main and regular servers follow independent phase type distributions with representations ($\boldsymbol{\alpha}$, U) and ($\boldsymbol{\beta}$, V) with number of phases 'p' and 'q', respectively. Note that $U^0 = -Ue$ and $V^0 = -Ve$. M denotes the upper bound of number of interruptions to the customer at the main server. Requirement of consultation follows Poisson process with rate θ_i, if there are i busy regular servers, where i = 1,2. Duration of consultation follows exponential distribution with parameter ξ.

Notations

In this sequel, we use the following notations.

(i) $\tilde{\boldsymbol{\alpha}} = \mathbf{e}'_{M+1}(1) \otimes \boldsymbol{\alpha}$

(ii) $D = \text{diag}(I \otimes \boldsymbol{\beta}, \boldsymbol{\beta} \otimes I)$, D= [D O]

(iii) E = diag(I⊗U,U⊗I), $E=\begin{bmatrix}E\\O\end{bmatrix}$, E = [E⊗I O]', E=diag(E⊗I, O), E = diag(E, O)

(iv) F = diag(I⊗U⊗ **β**, U⊗ **β**⊗I), F = diag(F, O), F = diag(I⊗U, U⊗I)

(v) $J = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$.

Consider the queueing model X = {X(t),t ≥ 0}, where

$X(t)=\{N(t), S(t), L_1(t), L_2(t), P_1(t), P_2(t), T(t), P_3(t)\}$.

Here

N(t) - the number of customers in the system

$P_i(t)$ - phase of the regular server R, i=1,2

T(t) - number of interruptions already befell to a customer at the main server

$P_3(t)$ - phase of the main server.

Here S(t) denotes the status of the servers at time t such that

S(t) = {
- $\tilde{0}$, if the regular server(s) is busy and main server is idle
- 0, if the main server is busy and regular server is busy or idle
- 1, if the main server is giving consultation only
- 2, if the main server is giving consultation with one interrupted customer at the main server
- 3, if the regular server is waiting for getting consultation after the present service at the main server
- 4, if the regular server is waiting for getting consultation after the service at the main server followed by the present interruption

The variable $L_1(t)$ appears only when N(t)=1 and S(t)= $\tilde{0}$ or N(t)=2 and S(t)=0.

$L_1(t)=\{1,2\}$ according to $\mathfrak{R}_1$ or $\mathfrak{R}_2$ is busy.

Now consider the variable $L_2(t)$.

If N(t) ≥ 1 and S(t)={1,2},

Then

$L_2(t)$= {
- 1 (*or* 2,) if $\mathfrak{R}_1$ is getting consultation and $\mathfrak{R}_2$ is busy or idle (or vice versa)
- 3 (*or* 4,) if both regular servers are in a queue for consultation with $\mathfrak{R}_1$ is getting consultation in the first place and $\mathfrak{R}_2$ in the second place (or vice versa).

If N(t) ≥ 1 and S(t) = 3, then $L_2(t)$ takes the same values {1,2,3,4} according to the above definition with `getting consultation' is replaced by `waiting to get consultation.'

If N(t) ≥ 3 and S(t)=4, then

$L_2(t)$=1 (or 2), if $\mathfrak{R}_1$ (or $\mathfrak{R}_2$) is waiting to get consultation after the present interruption followed by the service completion at the main server.

{X(t), t ≥ 0} is a CTMC with state space ψ = {0} ∪$\cup_{i=1}^{\infty}\psi(i)$.

The terms $\psi(i)$'s are defined as

$\psi(1)$ = {(1, 0, k, l): 0 ≤ k ≤ M }∪ {(1,$\tilde{0}$, j, l) ∪ (1,1,j,l): 1 ≤ j ≤ 2},

$\psi(2)$ = {(2, 0, j, l, k, n) : 0 ≤ k ≤ M, 1 ≤ j ≤ 2} ∪ {(2, $\tilde{0}$, l, m) ∪(2, 1, j, l, m): 1 ≤ j ≤ 4}∪{(2, 2, j, l, k, n): 1 ≤ j ≤ 2, 0 ≤ k ≤ M-1} ∪{(2, 3, j, l, n): 1 ≤ j ≤ 2},

and for i ≥ 3,

$\psi(i)$ = {(i, 0, l, m, k, n): 0 ≤ k ≤ M} ∪ {(i, 1, j, l, m): 1 ≤ j ≤ 4}
∪ {(i, 2, j, l, m, k, n): ≤ j ≤ 2,0 ≤ k ≤ M-1}∪ {(i, 2, j, l, m, k, n): 3 ≤ j ≤ 4, 0 ≤ k ≤ M-2} ∪{(I, 3, j, l, m, n): 1 ≤ j ≤ 4} ∪ {(i, 4, j, l, m, n): 1 ≤ j ≤ 2}, where 1 ≤ l, m ≤ q and 1 ≤ n ≤ p.

The infinitesimal generator matrix Q is given by

$$Q=\begin{bmatrix} -\lambda & D_0 & & & & & \\ B_1 & C_1 & D_1 & & & & \\ & B_2 & C_2 & D_2 & & & \\ & & B & A & A & & \\ & & & A & A & A & \\ & & & & \ddots & \ddots & \ddots \end{bmatrix} \quad (1)$$

Here A_0, A_1 and A_2 are square matrices of order K_3; C_1 and C_2 are square matrices of orders K_1 and K_2, respectively. D_0, D_1,D_2,B_1, B_2 and B_3 are matrices of orders 1 X K_1, K_1 X K_2, K_2 X K_3, K1 X 1, K_2 X K_1 and K_3 X K_2, respectively, where$K_1 = (M+1)p + 4q$, $K_2 = 4\,(M+1)pq + 5q^2$ and $K_3 = 5\,(M+1)pq^2 + 4q^2$.

The block matrices are defined as follows:

$$D_0 = \lambda\,[\tilde{\boldsymbol{\alpha}}\mathbf{0}], B_1 = \begin{bmatrix} \boldsymbol{e}_{M+1}\otimes U^0 \\ \mathbf{e}_2\otimes V^0 \\ O \end{bmatrix}, C_1 = \begin{bmatrix} I_{M+1}\otimes U & O \\ O & C_{11} \end{bmatrix} - \lambda I,$$

$$D_1 = [D_{11} \quad D_{12} \quad D_{13}]', \qquad B_2 = [B_{21} \quad B_{22} \quad B_{23}],$$

$$D_2 = \lambda[D_{12} \quad D_{22}]', \qquad B_3 = [B_{31} \quad B_{32} \quad B_{33}],$$

$$A_1 = \begin{bmatrix} A_{11} & & A_{12} \\ A_{13} & A_{14} & \\ A_{15} & & A_{16} \end{bmatrix} - \lambda\, I, \qquad A_2 = [A_{21} \quad B_{32} \quad A_{22}], \text{ where}$$

$$C_{11} = \text{diag}(I_2\otimes V, O) + \begin{bmatrix} -\theta_1 & \theta_1 \\ \xi & \xi \end{bmatrix}\otimes I_{2q}, \qquad D_{11} = \begin{bmatrix} \frac{1}{2}\boldsymbol{e}_2{}'\otimes\boldsymbol{\beta}\otimes I_{(M+1)p} & O \end{bmatrix}_{(M+1)p\times K_2},$$

$$D_{12} = [I_{2q}\otimes\tilde{\boldsymbol{\alpha}} \quad O]_{2q\times K_2}, \qquad D_{13} = [O \quad E \quad O]_{2q\times K_2},$$

$$B_{21} = \begin{bmatrix} \boldsymbol{e}_2\otimes V^0\otimes I_{p(M+1)} \\ O \end{bmatrix}_{K_2\times(M+1)p}, \qquad B_{22} = \begin{bmatrix} I_{2q}\otimes\boldsymbol{e}_{M+1}\otimes U^0 & \boldsymbol{e}_2{}'\otimes F & O \end{bmatrix}'_{K_2\times 2q},$$

$$B_{23} = \begin{bmatrix} O & F_1 & O & I_{2q}\otimes U^0 \end{bmatrix}'_{K_2\times 2q}, \qquad D_{21} = [E\otimes\boldsymbol{e}_2\otimes I_{(M+1)p} \quad O]_{2(M+1)pq\times K_3},$$

$D_{22} = [diag(I_{q^2}\otimes\tilde{\boldsymbol{\alpha}}, I_{4q^2}, E_1\otimes I_{pM},\ E_1\otimes I_p) \quad O]_{5q^2+2(M+1)pq\times K_3}$,

$B_{31}=\begin{bmatrix} \boldsymbol{e}_2{}'\otimes F \otimes I_{(M+1)p} & I_{q^2}\otimes\boldsymbol{e}_{M+1}\otimes U^0 \\ O & O \end{bmatrix}_{K_3\times 2(M+1)pq+q^2}$,

$B_{32} = [O \quad H_1 \quad O \quad I_{4q^2}\otimes U^0 \quad O]'_{K_{3\times 4q^2}}$, $B_{33} = \begin{bmatrix} O \\ diag\ (F_2\otimes I_p,\ F_1\otimes I_p\ ) \\ O \end{bmatrix}_{K_3\times 2(M+1)pq}$,

$A_{11}=$ V$\oplus$ V $\oplus I_{M+1}\otimes$ U - $2\theta_2 I_{(M+1)pq^2}$, $A_{12} = [\boldsymbol{e}_2{}'\otimes I_{q^2}\otimes\hat{\mathbf{I}}_M \quad O \quad \boldsymbol{e}_2{}'\otimes I_{q^2}\otimes\hat{\mathbf{e}}_M \quad O]$,

$A_{13}=\begin{bmatrix} \boldsymbol{e}_2\otimes I_{q^2}\otimes\tilde{\boldsymbol{\alpha}} \\ O \end{bmatrix}$, A_{14} = diag (H_2, O) $- \xi I_{4q^2}+\begin{bmatrix} -\theta_1 I_2 & \theta_1 I_2 \\ \xi J & O \end{bmatrix}\otimes I_{q^2}$, $A_{15}=\begin{bmatrix} \boldsymbol{e}_2\otimes I_{q^2}\otimes\dot{I}_M \\ O \end{bmatrix}\otimes I_p$,

$A_{16} =\begin{bmatrix} G_1 & & G_2 \\ & G_3 & \\ & G_4 & -\xi\ I_{2pq^2} \end{bmatrix}$, $A_{21} = \begin{bmatrix} V^0\otimes\boldsymbol{\beta}\ \oplus\ V^0\otimes\boldsymbol{\beta}\oplus\boldsymbol{e}_{M+1}\otimes U^0\otimes\boldsymbol{\alpha} \\ O \end{bmatrix}_{K_3\times(M+1)pq^2}$,

$A_{22} = \begin{bmatrix} O \\ diag(F_3\otimes I_p\ , F_4\otimes I_p, O) \end{bmatrix}_{K_3\times 4\,(M+1)pq^2}$.

Here

G_1 = diag $(H_2\otimes I_{(M+1)p}, O) - \xi\ I_{4q^2(2M-1)} + \begin{bmatrix} -\theta_1 I_{2Mq^2} & \theta_1 I_{2q^2}\otimes\hat{I}_{M_1} \\ \xi\ J\otimes I_{M_1\,pq^2} & O \end{bmatrix}\otimes I_p$,

$G_2 = \theta_1\begin{bmatrix} I_{2q^2}\otimes\hat{\mathbf{e}}_{M-1}\otimes I_p \\ O \end{bmatrix}$, G_3 = diag(H_2, O) $\otimes I_p$ - $I_{4q^2}\otimes$ U + $\begin{bmatrix} -\theta_1 I_{2pq^2} & \theta_1 I_{2pq^2} \\ O & O \end{bmatrix}$,

$G_4= \xi[J \quad O]\otimes I_{ab^2}$.

Steady state analysis[3]

In this section we perform the steady-state analysis of the queueing model under study. We first establish the stability condition of the queueing system.

3.1 Stability condition

Let π denotes the steady-state probability vector of the generator matrix

$A \ = A_0 + A_1 + A_2$.

That is, $\pi A = 0;\ \ \pi e = 1$. (2)

The LIQBD description of the model indicates that the queueing system is stable (see, Neuts [13]) if and only if

$$\pi A_0\boldsymbol{e} < \pi A_2\boldsymbol{e}. \quad (3)$$

That is, the rate of drift to the left has to be higher than that to the right. The vector π cannot be obtained explicitly in terms of the parameters of the model, and hence stability condition is known only implicitly. Define the traffic intensity ρ as

$$\rho = \frac{\pi A_0 \boldsymbol{e}}{\pi A_2 \boldsymbol{e}} \tag{4}$$

Note that the stability condition in (3) is equivalent to $\rho < 1$. We will discuss the impact of the input parameters of the model on the traffic intensity in Section 5.

3.2 Steady State Probability Vector

Since the model studied as a QBD process, its steady-state distribution has a matrix geometric solution under the stable condition. Assume that the stability condition is satisfied. Let $\boldsymbol{x}$denotes the steady state probability vector of the generator Q given in (1).

That is ,

$$\boldsymbol{x}Q = 0\ ;\ \ \boldsymbol{x}\boldsymbol{e} = 1. \tag{5}$$

Partitioning $\boldsymbol{x}$ **as**

$$\boldsymbol{x} = (\boldsymbol{x}_0,\ \boldsymbol{x}_1,\ \boldsymbol{x}_2,\ \ldots\ldots\ldots),$$

we see that, under the assumption that the stability condition (2) holds, the sub-vectors $\boldsymbol{x}_j$, j ≥ 4 are obtained as (see, Neuts[13])

$$\boldsymbol{x}_j = \boldsymbol{x}_1 R^{j-3},\ \ j \geq 4, \tag{6}$$

where$R$ is the minimal non-negative solution of the matrix equation

$$R^2 A_2 + RA_1 + A_0 = 0. \tag{7}$$

Knowing the matrix R, the vectors $\boldsymbol{x}_0$, $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are obtained by solving the boundary equations

$$\begin{aligned}
-\lambda \boldsymbol{x}_0 + \boldsymbol{x}_1 B_1 &= 0\\
\boldsymbol{x}_0 D_0 + \boldsymbol{x}_1 C_1 + \boldsymbol{x}_2 B_2 &= 0\\
\boldsymbol{x}_1 D_1 + \boldsymbol{x}_2 C_2 + \boldsymbol{x}_3 B_3 &= 0\\
\boldsymbol{x}_2 D_2 + \boldsymbol{x}_3 (A_1 + RA_2) &= 0
\end{aligned} \tag{8}$$

The normalizing condition of (5) is

$$\boldsymbol{x}_0 + \boldsymbol{x}_1 \boldsymbol{e} + \boldsymbol{x}_2 \boldsymbol{e} + \boldsymbol{x}_3 (I - R)^{-1} \boldsymbol{e} = 1. \tag{9}$$

Once the rate matrix R is obtained, the vector $\boldsymbol{x}$can be computed by exploiting the special structure of the coefficient matrices. We can use the iterative formulas (see Neuts [13]) $R_n = -A_0(A_1 + R_{n-1}A_2)^{-1}$, for $n \geq 1$, with an initial value R_0, which converges to R if $sp(R) < 1$.

Performance Measures [4]

In this section we list a number of key system performance measures to bring out the qualitative aspects of the model under study. These are listed below along with their formulae for computation. Towards this end, we further partition the vectors $\boldsymbol{x}_i, i \geq 1$ as $\boldsymbol{x}_1 = (\boldsymbol{x}_{10}, \ \boldsymbol{x}_{1\tilde{0}}, \ \boldsymbol{x}_{11}), \boldsymbol{x}_2 = (\boldsymbol{x}_{20}, \ \boldsymbol{x}_{2\tilde{0}}, \boldsymbol{x}_{21}, \ \boldsymbol{x}_{22}, \boldsymbol{x}_{23}),$

and $\boldsymbol{x}_i = (\boldsymbol{x}_{i0}, \ \boldsymbol{x}_{i1}, \ \boldsymbol{x}_{i2}, \boldsymbol{x}_{i3}, \boldsymbol{x}_{i4}),$ for $i \ \geq 3$.

Note that $\boldsymbol{x}_0$ is a scalar, $\boldsymbol{x}_{1\tilde{0}}, \boldsymbol{x}_{11}$ are vectors of dimension 2q; $\boldsymbol{x}_{i1,} i \geq 2$ is a vector of dimension $4q^2$; $\boldsymbol{x}_{10}, \ \boldsymbol{x}_{2\tilde{0}}, \boldsymbol{x}_{22}, \boldsymbol{x}_{23}$, $\boldsymbol{x}_{i0}, \ \boldsymbol{x}_{i2}, \ \boldsymbol{x}_{i3}, \boldsymbol{x}_{i4}$, for $i \ \geq 3$ are vectors of dimensions $(M+1)p$, $2(M+1)pq$, q^2, $2Mpq$, $2pq$, $(M+1)pq^2$, $2(2M-)pq^2$, $4pq^2$, $2pq^2$, respectively.

Now we compute some performance measures as follows:

1. Expected number of customers in the system
   $\text{ES} = \sum_{i=1}^{\infty} i \, x_i \, \boldsymbol{e}.$
2. Expected number of customers in the queue
   $\text{EQ} = \sum_{i=3}^{\infty} (i-2) x_{i1} \, \boldsymbol{e} + \sum_{i=4}^{\infty} (i-3) [\, x_{i0} \, \boldsymbol{e} + x_{i2} \boldsymbol{e} + x_{i3} \boldsymbol{e} + x_{i4} \boldsymbol{e}].$
3. Effective rate of interruption
   $\text{EI} = \theta_1 \sum_{j=0}^{M-1} x_{20j} \boldsymbol{e} + \theta_2 \sum_{i=3}^{\infty} \sum_{j=0}^{M-1} x_{i0j} \, \boldsymbol{e} + \theta_1 \sum_{i=3}^{M-1} (x_{i21} + x_{i22}) \boldsymbol{e}.$
4. Fraction of time the main server is idle
   $F_{mi} = x_0 + x_{1\tilde{0}} \boldsymbol{e} + x_{2\tilde{0}} \boldsymbol{e}.$
5. Fraction of time the main server is busy serving a customer
   $F_{mb} = x_{10} \, \boldsymbol{e} + \sum_{i=2}^{\infty} (x_{i0} \boldsymbol{e} + x_{i3} \boldsymbol{e}).$
6. Fraction of time main server is interrupted
   $F_{min} = \sum_{i=2}^{\infty} (x_{i2} \boldsymbol{e} + x_{i4} \boldsymbol{e}).$
7. Fraction of time the first regular server is idle
   $F_{ri} = x_0 + x_{10} \boldsymbol{e} + x_{1\tilde{0}2} \boldsymbol{e}.$
8. Fraction of time the first regular server is busy serving a customer
   $F_{rb} = x_{1\tilde{0}1} \boldsymbol{e} + x_{2\tilde{0}} \boldsymbol{e} + x_{212} \boldsymbol{e} + \sum_{i=3}^{\infty} (x_{i0} \boldsymbol{e} + x_{i12} \boldsymbol{e} + x_{i22} \boldsymbol{e} + x_{i32} \boldsymbol{e}).$
9. Fraction of time the first regular server is under consultation
   $F_{rc} = \sum_{i=1}^{\infty} x_{i11} \boldsymbol{e} + \sum_{i=2}^{\infty} x_{i21} \, \boldsymbol{e}.$
10. Fraction of time the first regular server is waiting to get consultation after the present service at the main server
    $F_{wcs} = \sum_{i=2}^{\infty} x_{i31} \boldsymbol{e}.$
11. Fraction of time first regular server is waiting to get consultation after the present consultation
    $F_{wcc} = \sum_{i=2}^{\infty} (x_{i13} \boldsymbol{e} + x_{123} \boldsymbol{e}).$
12. Fraction of time first regular server is waiting to get consultation after the present service at the main server and consultation to the regular server
    $F_{wcsc} = \sum_{i=3}^{\infty} x_{i41} \, \boldsymbol{e}.$

Numerical Examples [5]

In this section we examine the effect of λ, $\theta_1$and θ_2 on various performance measures.Choose the following data so that the stability condition ρ<1 is satisfied. Let U = $\begin{bmatrix} -9 & 3 \\ 2 & -8 \end{bmatrix}$; V = $\begin{bmatrix} -12 & 6 \\ 5 & -10 \end{bmatrix}$; **α** = [0.3 0.7]; **β** = [0.4 0.6]; ξ = 2; M = 3.

Table 1: Effect of λ on various performance measures
θ_1 = 3, θ_2 = 2

λ	1	1.5	2	1.5	3
ρ	0.2161	0.3242	0.4322	0.5403	0.6483
ES	1.1568	1.3054	1.6390	2.3727	3.9522
EQ	0.0157	0.0892	0.3201	0.9073	2.2726
EI	0.0432	0.0956	0.1665	0.2542	0.3568
F_{mi}	0.0121	0.0224	0.0317	0.0376	0.0386
F_{mb}	0.0254	0.0566	0.0969	0.1421	0.1871
F_{min}	0.0227	0.0519	0.0936	0.1486	0.2170

From table 1 we see that as λ increases the traffic intensity ρ increases as is to be expected. The system is fed with more customers and so more customers are accumulated in the system and in queue. So ES and EQ increase. The main server has to serve more customers which results in an increase in F_{mb}. As number of customers increases, effective rate of interruption to the main server EI and thus the fraction of time the main server stay in interrupted state F_{min} also increase.

Table 2: Effect of $\theta_1$on various performance measures
λ = 3, θ_2 = 2

θ_1	2	2.5	3	3.5	4
ρ	0.5407	0.5946	0.6483	0.7018	0.7549
ES	2.2804	2.9499	3.9522	5.4581	7.7758
EQ	0.8050	1.3847	2.2726	3.5974	5.4917
EI	0.2476	0.3029	0.3568	0.4078	0.4538
F_{mi}	0.0488	0.0439	0.0386	0.0329	0.0270
F_{mb}	0.2622	0.2453	0.2280	0.2107	0.1935
F_{min}	0.1514	0.1841	0.2170	0.2492	0.2790

We see from table 2 that as θ_1 increases, EI also grows faster since it depends directly on θ_1.$ This results in a hike in F_{min}. So the main server gets lesser time to serve customers which results in a decrease in F_{mb}. As F_{min} increases the main server gets lesser time to be idle and so F_{mi} decreases. As a whole, there is a rapid accumulation of customers in the system and in the queue, and hence ES and EQ increase. Since the effective service time (the sum of the time taken for actual service completion and intermediate consultations) increases, the traffic intensity ρ will increase.

Table 3: Effect of θ_2 on various performance measures
λ = 3, θ_1 = 2

$\boldsymbol{\theta_2}$	2	3	4	5	6
$\boldsymbol{\rho}$	0.5407	0.6258	0.6844	0.7271	0.7594
$\boldsymbol{ES}$	2.2804	2.7377	3.2558	3.8234	4.4300
$\boldsymbol{EQ}$	0.8050	1.1914	1.6399	2.1378	2.6717
$\boldsymbol{EI}$	0.2476	0.2881	0.3245	0.3568	0.3852
$\boldsymbol{F_{mi}}$	0.0488	0.0446	0.0408	0.0375	0.0346
$\boldsymbol{F_{mb}}$	0.2622	0.2479	0.2346	0.2224	0.2113
$\boldsymbol{F_{min}}$	0.1514	0.1844	0.2142	0.2406	0.2639

From table 3 we see that as θ_2 increases EI increases since EI depends directly on θ_2 and so F_{min} also increases. So the idle time F_{mi} of the main server decreases. Since the fraction of interrupted time of main server increases, the main server gets lesser time to serve customers and so F_{mb} decreases. The accumulation of customers increases, since the time for the service completion of customers increases. Thus both ES and EQ increase. Here also ρ increase.

Concluding remarks and suggestions for further study

In this paper we studied a three-server queueing model with consultations. Consultation is an important aspect which enhance the reliability of the services provided by the trainees by accepting timely advices and clarifications from the experienced servers. We can extend this model to introduce a super clock to measure the total duration of interruption to a customer at the main server. It will be interesting to deal with a three server queue with different arrival processes to the main and the regular servers. Two different queues can be maintained to the main and regular servers with a finite buffer at the main server. Then an optimum value of the buffer size can be evaluated.

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