NEW IDEALS CONTAINING THE KERNAL OF A FRAME HOMOMORPHISM

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Abstract: *In the background of point free topology, if f: L→ L is a frame homomorphism,the ideals [b]={ x ∈L : $\mathcal{E}_{f(x)} \cap \mathcal{E}_b = \Phi$ } corresponding to each b∈ L have been constructed. It is observed that for each b∈ L,[b] contains the ideal kerf ={ x∈ L : f(x)=0}. It is proved that the collection M={[b] ;b∈ L } is a complete meet semilattice under the partial order ⊆.*

Keywords: *Spectrum, Ideal, Kernal, semi lattice.*

INTRODUCTION [1]

It is well known that a topological space is a lattice of open sets. With the work of Marshall Stone on the topological representation of Boolean algebra and distributive lattices,the connection between topology and lattice theory began to be explored. Frame can be viewed as an extension of both topology and lattice theory. As a dual notion of category of frames,we have category of locale.

Johnstone in his paper "The point of pointless topology" expressed the complete lattice satisfying infinite distributive law as point free topology. After it most of the topological ideas have been studied in the frame background.

In this paper we have constructed the ideals [b] containing the kernel ker f of the given frame homomorphism f: L→ L. The collection M= {[b] : b∈ L } of ideals of L forms a complete meet semilattice with bottom element [1].

BASIC CONCEPTS [2]

2.1 Definition[2]

A poset L is a frame if and only if

i) Every subset of L has a join (least upper bound)

ii) Every finite subset has a meet (greatest lower bound)

iii) Binary meets distribute over arbitrary join. That is $a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\}$.

Since every finite subset of L has both join and meet,$\wedge\Phi$ and $\bigvee\Phi$ are elements of L.$\wedge\Phi$ is the greatest element of L and is denoted by Top or 1.$\bigvee\Phi$is the least element of L and is denoted by bottom or 0.The elements of frame are known as opens.

2.2 Examples [2]

1. Every finite distributive lattce is a frame.
2. For any set X, the power set P(X) is a frame, where meet is the intersection and join is the union of sets.

2.3 Definition [2]

A subset I of a frame L is said to be an ideal if

i) I is a sub-join-semi lattice of L; ie.$0\in I$ and $a\in I$,$b\in I$ imply $a\vee b\in I$;and

ii) I is a lower set;ie.$a\in I$ and $b\leq a$ imply $b\in I$

For any $a\in L$ the subset $\downarrow(a)=\{x : x\leq a\}$is an ideal of L and it is the smallest ideal containinga .It is called the principal ideal generated by a .

2.4 Definition [1]

We define frame homomorphism between the frames L and M as a map $h:L\to M$ which preserves all joins(including the bottom) and all finite meets (including the Top). The category **Frm** of frames has frames as objects and frame homomorphism as morphisms of the category. Category **Loc** of locales is opposite to the category of frames. When we consider the objects the category **Frm** and **Loc** are same. But it behaves differently when we consider morphisms. We can define a functor Ω from the category **Top** of topological spaces into the category **Frm** as follows. Ω sends each object X of **Top** to the frame Ω (X) of its open sets. If $f: X\to Y$ is a morphism in **Top**, that is f is a continuous function , then define $\Omega(f): \Omega(Y)\to\Omega(X)$ by $\Omega(f)(V)= f^{-1}(V)$.Then Ω is a contravarient functor from the category **Top** to the category **Frm**.

2.5 Definition

Let $f: L\to M$ be a frame homomorphism. Then the kernel of the homomorphism f is defined by $\ker f=\{x\in L : f(x)=0_M\}$.

2.6 Proposition

If $f: L\to M$ is a frame homomorphism, then kerf is an ideal in L.

Proof

Let $f: L\to M$ be a frame homomorphism. Then by definition $0_L\in\ker f$. Thus ker f is non empty.

Let $x,y\in\ker f$. Then we have $f(x)=0_M$ and $f(y)=0_M$.

Now $f(x\vee y)= f(x)\vee f(y)=0_M \vee 0_M = 0_M$. Thus $x\vee y\in\ker f$. Hence ker f is a sub join semi lattice.

Let $x\in\ker f$ and $y\leq x$. Since $y\leq x$, we have $y\wedge x= y$. Then

$f(y)= f(y\wedge x)= f(y)\wedge f(x)= f(y)\wedge 0_M = 0_M$. $f(y)=0_M$ implies $y\in\ker f$.

Thus by 2.3, ker f is an ideal in L.

2.7 Definition [1]

A subset S of frame L is said to be subframe of L if it is closed under all joins and finite meets. That is S itself a frame under the induced order of L.

Topology on a set X can be viewed as a subframe of P(X).This frame of opens is denoted by Ω(X) .Now the question is can we find a topology from a given frame. This question is answered using different methods. Here we use the concept of filter to answer the question.

2.8 Definition [1]

Let L be a frame. A subset F of L is said to be a filter if

i) F is closed under finite meet. That is if $a,b\in F$ then $a\wedge b\in F$

ii) F is an upper set. That is if $b \geq a$ for some $a \in F$, then $b \in F$

A filter F is said to be proper if $F \neq L$. A proper filter F is said to be prime if $b_1 \vee b_2 \in F \Rightarrow b_1 \in F$ or $b_2 \in F$. In a completely prime filter we require that this holds for any join, not only for finite ones. That is for any J and $b_i \in L$, $i \in J, \bigvee b_i \in F \Rightarrow b_i \in F$ for some $i \in J$. We use c.p filters to denote completely prime filters. A typical completely prime filter in $\Omega(X)$ is the system $U(x)=\{V \in \Omega(X) \; ; x \in V\}$ of all the open neighborhoods of a point x.

2.9 Spectrum [1]

For an element a of a locale L ,set $\mathcal{E}_a =\{F \text{ c.p filter in } L, a \in F\}$.
Then we have $\mathcal{E}_0 =\Phi, \mathcal{E}_1 =\{\text{all c.p filters}\}$
Now $F \in \mathcal{E}_{a \wedge b} \Leftrightarrow a \wedge b \in F \Leftrightarrow a \in F$ and $b \in F \Leftrightarrow F \in \mathcal{E}_a$ and $F \in \mathcal{E}_b \Leftrightarrow F \in \mathcal{E}_a \cap \mathcal{E}_b$.
Hence $\mathcal{E}_{a \wedge b} = \mathcal{E}_a \cap \mathcal{E}_b$.
Similarly $\mathcal{E}_{\bigvee b_i} = \cup \, \mathcal{E}_{b_i}$.
Thus $\mathcal{E}_a$ for $a \in L$, satisfies the properties of open sets in a topological space. Then using completely prime filters we define a topological space as follows.
Let $X=\{\text{all c.p filters}\}$, $\Omega(x)=\{\mathcal{E}_a \; , a \in L\}$.
Then $(X, \Omega(X))$ is called spectrum of L and is denoted by Sp(L).
If $f: L \to M$ is a localic map, we define $Sp(f): Sp(L) \to Sp(M)$ by $Sp(f)(F)=(f^*)^{-1}(F)$. Then Sp: Loc → Top is a functor and it is called spectrum functor. Spectrum functor is a covariant functor.

A frame L is spatial if it is isomorphic to the topology of its spectrum.

IDEALS [b] FOR b∈L[3]

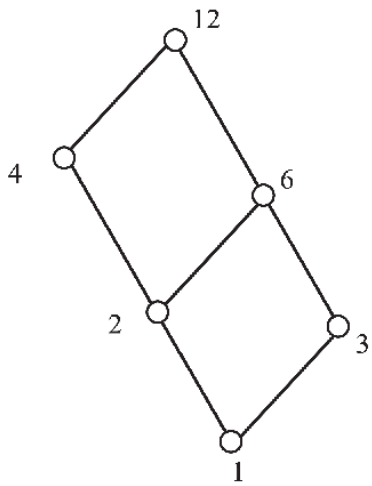
3.1 Definition

Let $f: L \to L$ be a frame homomorphism. For each $b \in L$, define
$[b]=\{x \in L : \mathcal{E}_{f(x)} \cap \mathcal{E}_b = \Phi \}$.

3.2 Example

Let the frame L be given as follows. Define $f: L \to L$ by

$$f(x)=\begin{cases} 1 & if\ x = 1 \\ 2 & if\ x = 2,3 \\ 3 & if\ x = 3 \\ 6 & if\ x = 6,12 \end{cases}$$



Then the completely prime filters of L are given by

F_1= {2,4,6,12} F_2= {3,6,12} F_3= {4,12} .

$\Sigma_1 =\Phi$, $\Sigma_2=\{F_1\}$, $\Sigma_3=\{F_2\}$, $\Sigma_4=\{F_1,F_3\}$, $\Sigma_6=\{F_1,F_2\}$, $\Sigma_{12}=\{F_1,F_2,F_3\}$.

The spectrum Sp(L) is given by

Sp(L)= $\{F_1,F_2,F_3\}$ with $\Omega(Sp(L))=\{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_6, \Sigma_{12}\}$.

Now

[3]={x∈ L: $\mathcal{E}_{f(x)}\cap\mathcal{E}_3 =\Phi$ }={1,2,4}

[4]={x∈ L: $\mathcal{E}_{f(x)}\cap\mathcal{E}_4 =\Phi$ }={1,3 }

[2]={x∈ L: $\mathcal{E}_{f(x)}\cap\mathcal{E}_2 =\Phi$ }={1,3}

[6]={x∈ L: $\mathcal{E}_{f(x)}\cap\mathcal{E}_6 =\Phi$ }={1}

[1]={x∈ L: $\mathcal{E}_{f(x)}\cap\mathcal{E}_1 =\Phi$ }=L

[12]={x∈ L: $\mathcal{E}_{f(x)}\cap\mathcal{E}_{12} =\Phi$ }={1 }

3.3 Proposition

Let f:L→ L be a frame homomorphism. For each b ∈ L, [b] is an ideal of L.

Proof

Since f is a frame homomorphism, we have f(0)=0.Hence 0∈ [b].

Let x,y∈ [b]. Then we have $\mathcal{E}_{f(x)}\cap\mathcal{E}_b=\Phi$ and $\mathcal{E}_{f(y)}\cap\mathcal{E}_b=\Phi$.

Thus $\mathcal{E}_{f(x\vee y)}\cap\mathcal{E}_b=\mathcal{E}_b\cap(\mathcal{E}_{f(x)}\cup\mathcal{E}_{f(y)})=(\mathcal{E}_{f(x)}\cap\mathcal{E}_b)\cup(\mathcal{E}_{f(y)}\cap\mathcal{E}_b)=\Phi$, which implies x∨y∈ [b].

Hence [b] is a join semilattice.

Let x∈ [b] and y∈ L be such that y≤ x.

Since x∈[b],we have $\mathcal{E}_{f(x)}\cap\mathcal{E}_b=\Phi$. Since y≤ x, f(y)≤ f(x) and hence $\mathcal{E}_{f(y)}\subseteq\mathcal{E}_{f(x)}$.

$\mathcal{E}_{f(x)}\cap\mathcal{E}_b=\Phi$ implies $\mathcal{E}_{f(y)}\cap\mathcal{E}_b=\Phi$. Thus y∈ [b].

Hence [b] is an ideal on L.

3.4 Proposition

i. If a≤ b in L, then [b]⊆ [a].

ii. The kernel kerf ⊆ [b] for all b∈ L.

iii.[0]= L

iv. If L is a spatial locale, then [1]=kerf.

Proof

i. Let $a \leq b$ in L and $x \in [b]$. Since $x \in [b]$, we have $\mathcal{E}_{f(x)} \cap \mathcal{E}_b = \Phi$.
$a \leq b$ implies $\mathcal{E}_a \subseteq \mathcal{E}_b$. Hence $\mathcal{E}_{f(x)} \cap \mathcal{E}_a \subseteq \mathcal{E}_{f(x)} \cap \mathcal{E}_b = \Phi$. That is $\mathcal{E}_{f(x)} \cap \mathcal{E}_a = \Phi$. Thus $x \in [a]$.
Hence $[b] \subseteq [a]$.

ii. Let $x \in \ker f$. Then by 2.5, $f(x)=0$. So $\mathcal{E}_{f(x)} = \mathcal{E}_0 = \Phi$, by 2.9.
Then $\mathcal{E}_{f(x)} \cap \mathcal{E}_b = \mathcal{E}_0 \cap \mathcal{E}_b = \Phi \cap \mathcal{E}_b = \Phi$, for all $b \in L$. Hence $\ker f \subseteq [b]$, for all $b \in L$.

iii. By definition $[0] = \{ x \in L;\ \mathcal{E}_{f(x)} \cap \mathcal{E}_0 = \Phi \}$. Since $\mathcal{E}_0 = \Phi$, we have $\mathcal{E}_{f(x)} \cap \mathcal{E}_0 = \Phi$ for all $x \in L$. Hence $[0]=L$.

iv. Also $[1] = \{ x \in L;\ \mathcal{E}_{f(x)} \cap \mathcal{E}_1 = \Phi \}$. Since $\mathcal{E}_1 = \{$all completely prime filters of L$\}$,
$\mathcal{E}_{f(x)} \cap \mathcal{E}_1 = \Phi$ if and only if $\mathcal{E}_{f(x)} = \Phi$. Since L is spatial $\mathcal{E}_{f(x)} = \Phi$ if and only if $f(x)=0$.
Hence $\mathcal{E}_{f(x)} \cap \mathcal{E}_1 = \Phi$ if and only if $x \in \ker f$.
Thus $[1] = \ker f$.

3.5 Proposition

The collection $M = \{[b] : b \in L\}$ is a complete meet semilattice with respect to the partial order $\subseteq$.

Proof

Order M by subset ordering. Then $(M, \subseteq)$ is a partially ordered set.
Let $x \in \cap [b_i]$. Then we have $\mathcal{E}_{f(x)} \cap \mathcal{E}_{b_i} = \Phi$ for all $i \in I$.
Then $\mathcal{E}_{f(x)} \cap \mathcal{E}_{\vee b_i} = \mathcal{E}_{f(x)} \cap (\cup \mathcal{E}_{b_i}) = \cup (\mathcal{E}_{f(x)} \cap \mathcal{E}_{b_i}) = \Phi$, which implies $x \in [\vee b_i]$.
Hence $\cap [b_i] \subseteq [\vee b_i]$.
Also by 3.4 (i), $[\vee b_i] \subseteq [b_i]$ for all $i \in I$..Thus $[\vee b_i] \subseteq \cap [b_i]$
Hence $[\vee b_i] = \cap [b_i]$. Thus M is a complete meet semi lattice with bottom element [1].

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